Separable Isogenies - Talk 13 in Jacobians of Curves

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Throughout, k will be a fixed algebraically closed field, and all schemes taken over k. In today's talk we'll be exploring the following notion:

Definition 1. *Let X and Y be abelian varieties over k. An* ***isogeny*** *$f : X \to Y$ between X and Y is a morphism of abelian varieties which is surjective and of finite kernel.*

To be precise, the kernel $\ker f$ of the morphism $X \to Y$ is just given by the fibre product

$$\begin{array}{ccc} \ker(f) & \longrightarrow & \operatorname{Spec} k \cong \{0\} \subset X \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

If X is simply an abelian group, then an "isogeny" out of X is completely determined by its kernel, since the surjectivity of f in the above definition identifies Y with the quotient $X/\ker(f)$. Our main end today will be do understand how to translate this statement to the setting of abelian varieties - although this might initially seem like a dull objective, it'll force us to consider quotients of varieties by group actions (since we can't really make sense of the object "$X/\ker(f)$" otherwise) and a particular case of faithfully flat descent.

1 Quotients of varieties by finite group-actions

We'll start off by refining a theorem we discussed in Talk 5:

Theorem 1. *Let X be an variety over k, G a finite subgroup in $\operatorname{Aut}_{\mathrm{Sch}/k}(X)$. Suppose that for every $x \in X$ there exists an affine open $U \subset X$ containing the finitely many points in the orbit Gx (from now on this property will be referred to as* ***affine-admissibility*** *- this is* **not** *standard practice). Then there exists a variety Y over k together with a map $\pi : X \to Y$ satisfying:*

1. *on underlying topological spaces, $|Y|$ is given by the quotient topology $|X|/|G|$ and $|\pi| : |X| \to |Y|$ is the quotient map,*
2. *the structure sheaf $\mathcal{O}_Y$ on Y is given by the subsheaf of G-invariants of the sheaf $\pi_*\mathcal{O}_X$ on Y.*

Furthermore, π is a finite, surjective and separable morphism of varieties.

Proof. Y's uniqueness imposes that the above construction is local on X, meaning that if we construct $U//G$ as above for every G-invariant open subset $U \subset X$ (such open sets constitute a cover for X by hypothesis) then these automatically satisfy the cocycle condition and they must glue to form the "global" quotient $X//G$.
If $X \cong \operatorname{Spec} A$ is affine then we're forced to set $Y = \operatorname{Spec} A^G$ where $A^G \subset A$ is the subring of G-invariants, $X \to Y$ the morphism induced by the inclusion $A^G \hookrightarrow A$.
All the properties π is claimed to satisfy are also local on X, so once again it's sufficient to argue each of them in this particular case. If A is generated over k by elements $a_1, \ldots, a_n \in A$, and γ is the order of G, then the $n \cdot \gamma$ elements

$$\begin{gathered} \sigma_1(G \cdot a_1), \ldots, \sigma_\gamma(G \cdot a_1), \\ \ldots \\ \sigma_1(G \cdot a_n), \ldots, \sigma_\gamma(G \cdot a_n), \end{gathered}$$

- $\sigma_1, \ldots, \sigma_\gamma$ being the symmetric polynomials in γ variables - all lie in A^G by construction, and A is integral over the k-algebra B that these generate, because each of the elements in the orbit $G \cdot a_i$ is integral over $k[\sigma_1(G \cdot a_i), \ldots, \sigma_\gamma(G \cdot a_i)]$. This shows that A is finite over A^G, because A/B is finite and B is evidently Noetherian. From this we get that A^G is a k-algebra of finite type over k: if $y_1, \ldots, y_m \in A$ generate A as an A^G-module, then we can consider the relative structure constants

$$y_i \cdot y_j = \sum_k b_k^{i,j} \cdot y_k, \ \ b_k^{i,j} \in A^G.$$

If we also express $a_1, \dots, a_n$ in terms of $y_1, \dots, y_m$'s

$$a_i = \sum_k b_k^i \cdot y_k, \quad b_k^i \in A^G$$

then we may define B_0 to be the k-subalgebra generated by elements $\{b_k^{i,j}, b_k^i\}_{i,j}$. Evidently B_0 is contained in A^G, and expressing any element $x \in A^G \subset A$ in terms of $a_1, \dots, a_n$ and using the above expressions it follows that x can be expressed as a polynomial in the $b_k^{i,j}, b_k^i$'s.
We've thus shown that Y is a variety over k and that the map $X \to Y$ is finite. Since the fraction field of A^G is given by the G-invariants of the fraction field of A, it follows that the extension $\kappa(X)/\kappa(Y)$ is Galois and thus separable.
If x and x' are two closed points in X which have distinct orbits Gx and Gx' then for every $g \in G$ we must have that $\mathfrak{m}_{g \cdot x'}$ is not contained in the union $\bigcup_{g \in G} \mathfrak{m}_{g \cdot x}$ (by prime avoidance and maximality). Thus there exists some element $f_g \in A$ such that $f_g \in \mathfrak{m}_{g \cdot x'}$ but $f_g \notin \bigcup_g \mathfrak{m}_{g \cdot x} \implies f := \prod_g f_g \in \bigcap_g \mathfrak{m}_{g \cdot x'} \setminus \bigcup_g \mathfrak{m}_{g \cdot x}$. Since the prior intersection and union are G-invariant subsets of A, if we take the product $\prod_g g \cdot f$ of the orbit of f under G then we get a G-invariant element $Gf \in A^G$ which lies in the contraction (i.e. the image under $X \to Y$) of $\mathfrak{m}_{x'}$, but not in the contraction of $\mathfrak{m}_x \implies \pi(x) \neq \pi(x')$.
Conversely, for any $g \in G$ we have

$$\mathfrak{m}_x^c = \mathfrak{m}_x \cap A^G = \underbrace{g^{-1}(\mathfrak{m}_x)}_{=\mathfrak{m}_{g \cdot x}} \cap A^G = \mathfrak{m}_{g \cdot x}^c.$$

By distinguishing points by their height and using the going up theorem one may conclude that $\pi(x) = \pi(x') \iff Gx = Gx'$ for general points x and x' using the same argument. Whence π is the quotient map from $|X|$ to $|X|/|G|$ as stated, as it equals the set-theoretic quotient and π is closed on underlying topological spaces since it is specialising. ∎

Note that for quasi-projective varieties - namely, varieties admitting an ample line bundle $\mathscr{L} \in \mathrm{Pic}(X)$, as discussed in the lectures - the requirement that every orbit of G's action is covered by some affine open becomes vacuous as every finite set of points of X is contained in some affine open of the form $D(s), s \in H^0(X, \mathscr{L})$ (cfr. Exercise Sheet 1 AG2).

If G's action on X is *free* (meaning that for every k-scheme T the action on T-valued points $G \times X(T) \to X(T)$ is free) then the fibres of the map $\pi : X \twoheadrightarrow X//G$ don't have any "multiplicities"; this heuristic allows us to deduce algebro-geometric regularity properties π must satisfy.

Proposition 1. *Let X be a variety over k and G a finite group of automorphisms of X such that its action on X is **free** and affine-admissible. Then the quotient map $\pi : X \twoheadrightarrow X//G$ is étale.*

Proof. Since the family of étale maps satisfies the LOCS property, it'll be sufficient to argue the result for the case in which $X = \operatorname{Spec} A$ is affine, and $X//G = \operatorname{Spec} A^G$ is the space corresponding to the subring of G-invariants. Recall that (cfr. Proposition 3.26, chap. 4, [6]) since $X//G$ is locally Noetherian and $X \to X//G$ is of finite type, we may use the referenced criterion for étale morphisms and argue that for every closed point $x \in \operatorname{Spec} A$ and its contraction $\pi(x) \in \operatorname{Spec} A^G$, we have that the induced morphism at completions

$$\widehat{A^G}_{\mathfrak{m}_x^c} = \widehat{A^G}_{\mathfrak{m}_{\pi(x)}} \longrightarrow \widehat{A}_{\mathfrak{m}_x}$$

is an isomorphism (note that we are using that k is algebraically closed quite crucially here). Denote by $\widehat{A}$ the completion of A with respect to the ideal $\mathfrak{m}_x^c = \mathfrak{m}_{\pi(x)} \subset A^G$ thought of as an A^G-module, and since A^G is Noetherian and A is finite over A^G, we have an isomorphism of A^G-modules

$$\widehat{A} \cong \widehat{A^G}_{\mathfrak{m}} \otimes_{A^G} A.$$

Furthermore, since the fibre of the point $\mathfrak{m}_{\pi(x)} \in \operatorname{Spec} A^G$ is the orbit $G \cdot x = \{\mathfrak{m}_{g \cdot x} \mid g \in G\} \subset \operatorname{Spec} A$, we have that $\widehat{A}$ equals the completion of A with respect to $\prod_{g \in G} \mathfrak{m}_{g \cdot x}$, because $(\mathfrak{m}_x^c)^e$ contains some product of said ideal. This implies that each quotient

$$A / \prod_{g \in G} \mathfrak{m}_{g \cdot x}^n \cong \prod_{g \in G} A/\mathfrak{m}_{g \cdot x}^n$$

decomposes as a product, by the Chinese Remainder Theorem; hence taking colimits over n yields $\widehat{A} \cong \prod_{g \in G} \widehat{A}_{\mathfrak{m}_{g \cdot x}}$.

We have an exact sequence of A^G-modules

$$0 \to A^G \to A \longrightarrow \bigoplus_{g \in G} A, \qquad a \longmapsto (a - g \cdot a)_g$$

which yields, upon applying the exact functor $\widehat{A^G}_{\mathfrak{m}_{\pi(x)}} \otimes_{A^G} -$,

$$0 \to \widehat{A^G}_{\mathfrak{m}_{\pi(x)}} \to \widehat{A} \longrightarrow \bigoplus_{g \in G} \widehat{A}. \qquad a \longmapsto (a - g \cdot a)_g$$

and this shows that $\widehat{A^G}_{\mathfrak{m}_{\pi(x)}}$ is precisely the submodule of G-invariants in $\widehat{A}$. However, G's action on $\widehat{A}$ has been made explicit and easily described in the preceding discussion: through the natural isomorphism of A^G-modules $\widehat{A} \cong \bigoplus_{g \in G} \widehat{A}_{\mathfrak{m}_{g \cdot x}} \cong \bigoplus_{g \in G} \widehat{A}_{\mathfrak{m}_x}$ we see that G simply permutes terms in this direct sum. The G-invariants are thus isomorphic to $\widehat{A}_{\mathfrak{m}_x}$, since these can be identified with tuples all of whose components are equal and in $\widehat{A}_{\mathfrak{m}_x}$ - the fact that G acts freely on X implies that G's permutation action here indeed permutes *all* components. ■

The previous proof contains a technique which we'll encounter later as well, which is rather handy in some situations: note that, in relating the étaleness of π to a property about completions, we managed to turn G's non-trivial action on A to a *permutation* action which shuffles components in a direct product.

Example: Suppose k's characteristic isn't p and let $\zeta \in k$ be a primitive p-th root of unity. The cyclic group $C_p \cong \langle \zeta \rangle$ acts on $X = \mathbb{A}^1_k$ by the ring map

$$\zeta^i \in \mathbb{Z}/(p), \;\; \zeta^i : k[X] \to k[X], \qquad X \mapsto \zeta^i X$$

If $k = \mathbb{C}, X = \mathbb{A}^1_{\mathbb{C}}$, then this action can be though of as a rotation of $2\pi i/p$ radians around the origin in the complex plane (note that the 'i' here is just an index). The quotient $X//C_p$ is the spectrum of the ring $k[X^p]$, and the map $k[X^p] \hookrightarrow k[X]$ defines a morphism of schemes

$$\pi : X = \mathbb{A}^1_k \to X//G \cong \mathbb{A}^1_k$$

which isn't étale, as it presents an unramified point at the origin. Note that in this case G's action isn't simply transitive as every element in C_p fixes the origin.

2 Descent along torsors

Our main aim is now to relate the category of sheaves on X to that of its quotient $X//G$; this analysis will be in the form of *descent theory*: let Y be an arbitrary scheme and suppose $Y = \bigcup_i U_i$ is an open cover, and we denote by $U_{i_1,\dots,i_n}$ the intersection $U_{i_1} \cap \cdots \cap U_{i_n}$. Last term we discussed the equivalence of categories

$$\underline{\mathrm{QCoh}}_Y \xrightarrow{\cong} \{(\mathcal{F}_i\,, \phi_{i,j})_{i,j} \mid \mathcal{F}_i \in \underline{\mathrm{QCoh}}_{U_i},\;\; \phi_{i,j} : \mathcal{F}_i\,|_{U_{i,j}} \xrightarrow{\cong} \mathcal{F}_j\,|_{U_{i,j}} \;\text{ satisfying the cocycle condition}\},$$

where we say the collection of isomorphisms $(\phi_{i,j} : \mathcal{F}_i\,|_{U_{i,j}} \to \mathcal{F}_j\,|_{U_{i,j}})_{i,j}$ satisfies the cocycle condition if the following diagram is commutative, for every triple of indices i, j, k,

$$\begin{tikzcd} \mathcal{F}_i\,|_{U_{i,j,k}} \arrow[rr, "\phi_{i,j}|_{U_{i,j,k}}"] \arrow[dr, "\phi_{i,k}|_{U_{i,j,k}}"'] & & \mathcal{F}_j\,|_{U_{i,j,k}} \\ & \mathcal{F}_k\,|_{U_{i,j,k}} \arrow[ur, "\phi_{k,j}|_{U_{i,j,k}}"'] & \end{tikzcd}$$

This equivalence in some sense is trying to suggest that quasi-coherent modules on the disjoint union of open subsets $\coprod_i U_i$ (i.e. collections of quasi-coherent modules, one over each open subset from our cover) endowed with **coherent** gluing data, yield a unique module on Y which restricts to the given ones on each open from the cover $\bigcup_i U_i$.

We're interested in restating the above equivalence of categories in a way which doesn't really rely as much on the fact that each of the elements in the open cover admit an open immersion into Y, since this will provide a broader view on the mathematics behind this equivalence of categories.

Each of the intersections $U_{i,j}$ are canonically identified with the fibre product $U_i \times_Y U_j$, and the fibre product of $X = \coprod_i U_i$ with itself over Y is a disjoint union

$$X \times_Y X = \coprod_{i,j} U_i \times_Y U_j \cong \coprod_{i,j} U_i \cap U_j.$$

If we consider the two projection maps

$$\begin{tikzcd} X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

then the restrictions $\mathscr{F}_i\,|_{U_{i,j}}$ can be seen as the pullbacks along these projections

$$(\mathscr{F}_i\,|_{U_{i,j}})_{i,j} = p_1^*(\mathscr{F}_i)_i\,,\ \ (\mathscr{F}_j\,|_{U_{i,j}})_{i,j} = p_2^*(\mathscr{F}_j)_j \in \underline{\mathrm{QCoh}}_{X\times_Y X} = \prod_{i,j} \underline{\mathrm{QCoh}}_{U_i \cap U_j};$$

so the collections of isomorphisms $(\phi_{i,j})_{i,j}$ become an isomorphism of quasi-coherent modules over $X \times_Y X$

$$\phi := (\phi_{i,j})_{i,j} : p_1^*(\mathscr{F}_i)_i \xrightarrow{\cong} p_2^*(\mathscr{F}_j)_j.$$

The cocycle condition can be reinterpreted in a similar spirit as the commutativity of the triangle

$$\begin{tikzcd} q_1^*(\mathscr{F}_i)_i \arrow[rr, "q_{12}^*\phi"] \arrow[dr, "q_{13}^*\phi"'] & & q_2^*(\mathscr{F}_i)_i \arrow[dl, "q_{23}^*\phi"] \\ & q_3^*(\mathscr{F}_i)_i & \end{tikzcd}$$

Where $q_1, q_2, q_3, q_{12}, q_{13}, q_{23}$ are the corresponding projections from the triple fibre product $X \times_Y X \times_Y X$ (which, following the previous though process, corresponds to the space given by the disjoint union of all triple intersections from the considered open cover).

Definition 2. *Let $X \xrightarrow{f} Y$ be a scheme over Y. The* ***category of descent datum*** $\mathrm{Des}(X \xrightarrow{f} Y)$ *attached to f is the category of tuples $(\mathscr{F}, \phi)$ where $\mathscr{F} \in \underline{\mathrm{QCoh}}_X$ and $\phi : p_1^*\mathscr{F} \to p_2^*\mathscr{F}$ is an isomorphism satisfying the "cocycle condition" -*

$$\begin{tikzcd} q_1^*\mathscr{F} \arrow[rr, "q_{12}^*\phi"] \arrow[dr, "q_{13}^*\phi"'] & & q_2^*\mathscr{F} \arrow[dl, "q_{23}^*\phi"] \\ & q_3^*\mathscr{F} & \end{tikzcd}$$

is a commutative diagram of morphisms of sheaves on $X \times_Y X \times_Y X$.

Note that there always is a functor $\underline{\mathrm{QCoh}}_Y \to \mathrm{Des}(X \xrightarrow{f} Y)$, given by simply mapping $\mathscr{F} \in \underline{\mathrm{QCoh}}_Y$ to its pullback $f^*\mathscr{F}$ together with the identity isomorphism $p_1^* f^*\mathscr{F} = (f \circ p_1)^*\mathscr{F} \longrightarrow (f \circ p_2)^*\mathscr{F} = p_2^* f^*\mathscr{F}$.

What's remarkable is that the *only* properties that the morphism $\coprod_i U_i \to Y$ enjoys as to guarantee the equivalence of categories

$$\underline{\mathrm{QCoh}}_Y \xrightarrow{\cong} \mathrm{Des}(X \xrightarrow{f} Y)$$

can be described in the following result.

Theorem 2. *Suppose $f : X \to Y$ is faithfully flat and quasi compact (often abbreviated as fpqc). Then $\underline{\mathrm{QCoh}}_Y \to \mathrm{Des}(X \xrightarrow{f} Y)$ is an equivalence of categories.*

Proof. Can be found in [4], page 9. ■

In the situation of Proposition 1, we have that $\pi : X \twoheadrightarrow X//G$ is faithfully flat and quasi compact, since it's affine; the "fpqc descent" theorem applies, so we may study quasi-coherent sheaves on the quotient $\underline{\mathrm{QCoh}}_{X//G}$ by means of the descent datum category $\mathrm{Des}(X \xrightarrow{\pi} X//G)$.

It's natural to ask if it's possible to interpret the category $\mathrm{Des}(X \xrightarrow{\pi} X//G)$ under a different light, in this particular situation. This requires us to think of G as being enriched with slightly more structure than that of a group: we give G the structure of a k-*scheme*, by simply setting

$$G := \coprod_{g \in G} \operatorname{Spec} k$$

and thinking of G as a group scheme over k, together with a morphism of schemes $\mu : G \times_k X \to X$ which is defined via its functor of points. In such a way, the product $G \times_k X$ has a geometric structure and it thus makes sense to consider quasi-coherent sheaves on it.

Definition 3. *Suppose X is a variety over k and $G \leq \mathrm{Aut}_{\mathrm{Sch}\,/k}(X)$ is a finite group defining an affine-admissible free action on X, and fix $\mathcal{F} \in \underline{\mathrm{QCoh}}_X$. Denote by μ the "action" map and by p the projection onto X*

$$\begin{tikzcd} G \times_k X \arrow[r, "\mu"] \arrow[d, "p"'] & X \\ X & \end{tikzcd}$$

*A **lifting** of X's G-action to $\mathcal{F}$ is an isomorphism of sheaves on $G \times_k X$*

$$\phi : \mu^*\mathcal{F} \xrightarrow{\cong} p^*\mathcal{F}$$

satisfying the "associativity condition" - the diagram of morphisms of sheaves on $G \times_k G \times_k X$

$$\begin{tikzcd} (\mathrm{id}_G, \mu)^*\mu^*\mathcal{F} \arrow[rr, "(\mathrm{id}_G\ \ \mu)^*\phi"] \arrow[dr, "(m\ \ \mathrm{id}_X)^*\phi"'] & & p_{23}^*\mu^*\mathcal{F} \arrow[dl, "p_{23}^*\phi"] \\ & p_3^*\mathcal{F} & \end{tikzcd}$$

*A quasi-coherent sheaf $\mathcal{F} \in \underline{\mathrm{QCoh}}_X$ together with a lifting ϕ is called a G-**quasi-coherent sheaf on** X, and the category of such will be denoted by $\underline{\mathrm{QCoh}}_X^G$.*

Since G's action in the above definition is *free*, we actually have an isomorphism (as schemes!)

$$X \times_{X//G} X \cong G \times_k X$$

as can be checked easily by looking at the functors of points these spaces represent. Using this isomorphism the category of descent datum $\mathrm{Des}(X \xrightarrow{\pi} X//G)$ becomes equivalent to that of sheaves on X with a fixed lifting of X's action $\phi : \mu^*\mathcal{F} \to p^*\mathcal{F}$.
If $\mathcal{F} \in \underline{\mathrm{QCoh}}_{X//G}$ then $\pi^*\mathcal{F}$ has a lifting of X's G-action: we have a commutative diagram, for every $g \in G$,

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\ & X//G & \end{tikzcd}$$

which yields a isomorphism of sheaves $\phi_g : \pi^*\mathcal{F} \to g^*\pi^*\mathcal{F}$ and taking these component-wise provides an isomorphism $(\phi_{g^{-1}})_g : p^*\mathcal{F} \to \mu^*\mathcal{F}$ (note that $G \times_k X \cong \coprod_g X$). In light of the previous discussion, we may now prove our descent theorem for the morphism $\pi : X \to X//G$.

Proposition 2. *Let G act freely on X. The functor $\mathcal{F} \in \underline{\mathrm{QCoh}}_{X//G} \longmapsto \pi^*\mathcal{F} \in \underline{\mathrm{QCoh}}_X^G$ is an equivalence of categories between the category of quasi-coherent sheaves on $X//G$ and that of G-quasi-coherent sheaves on X. The quasi-inverse is given by the functor*

$$(\mathcal{G}, \phi) \in \underline{\mathrm{QCoh}}_X^G \longmapsto \pi_*(\mathcal{G})^G := p_*\,\mathrm{eq}(p^*\mathcal{F} \underset{id}{\overset{\phi}{\rightrightarrows}} \mu^*\mathcal{F}).$$

Proof. Since the quasi-inverse is explicit, we shall show that the natural morphisms

$$\mathcal{F} \to \pi_*(\pi^*\mathcal{F})^G, \ \ \mathcal{F} \in \underline{\mathrm{QCoh}}_{X//G}$$
$$\pi^*(\pi_*(\mathcal{G})^G) \to \mathcal{G}, \ \ \mathcal{G} \in \underline{\mathrm{QCoh}}_X^G$$

are isomorphisms of sheaves - notice that the latter is well-defined since, as discussed, the pullback of a quasi-coherent sheaf on $X//G$ has a canonical lifting of the G-action. As morphisms of sheaves, it'll be sufficient to prove they're isomorphisms locally on X and $X//G$, by restricting to a G-invariant affine open $\mathrm{Spec}\, A$ on X. The two maps above become

$$s_M : M \longrightarrow (M \otimes_{A^G} A)^G, \ \ M \in \underline{\mathrm{Mod}}_{A^G}$$
$$t_M : N^G \otimes_{A^G} A \to N, \ \ N \in \underline{\mathrm{Mod}}_A^G$$

where $\underline{\mathrm{Mod}}_A^G$ is the category of A-modules with a compatible G-action. The nice thing in this setting is that the composition of morphisms

$$M \in \underline{\mathrm{Mod}}_{A^G}, \quad A \otimes_{A^G} M \xrightarrow{1 \otimes s_M} (M \otimes_{A^G} A)^G \otimes_{A^G} A \xrightarrow{t_{M \otimes_{A^G} A}} M \otimes_{A^G} A$$

is an isomorphism, and since $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is faithfully flat, it follows that $1 \otimes s_M$ is an isomorphism if and only if s_M is $\implies$ it'll be sufficient to argue that t_N is an isomorphism for every $N \in \underline{\mathrm{Mod}}_A^G$. We now use the technique described in the remark after the previous proposition 1. If A is isomorphic to a product of the form $\prod_{g \in G} A^G$ as in the previous proof on the completed stalks, then t_N is certainly an isomorphism, since G's action on these factors is simply transitive. However since t_N is an isomorphism if and only if it's an isomorphism at each completion at the closed points in $\operatorname{Spec} A$ (just because $\operatorname{Spec} R^\wedge \to \operatorname{Spec} R$ is faithfully flat for any Noetherian *local* ring R) we get that this particular case implies the general result. ■

3 Characters and line bundles

We now specialise our understanding of the behaviour of $\pi^* : \operatorname{Pic}(X//G) \to \operatorname{Pic}(X)$ to the specific case in which X is a proper variety over k, getting us a step closer to the setting of abelian varieties.

Definition 4. *Let G be any group. The group of of homomorphisms $\widehat{G} := \operatorname{Hom}(G, k^\times)$ is called the group of **characters** of G. Suppose X and G are as in the previous section. For every character $\alpha \in \widehat{G}$ denote by $\mathscr{L}_\alpha$ the quasi-coherent sheaf on $X//G$ defined by*

$$U \subseteq X//G \longmapsto \mathscr{L}_\alpha(U) := \{a \in \pi_*(\mathscr{O}_X) \mid g \cdot a = \alpha(g)a, \text{ for all } g \in G\} \subset \pi_*(\mathscr{O}_X)(U).$$

The following proposition illustrates the relevance of G's characters in this setting.

Proposition 3. *Suppose $X \to \operatorname{Spec} k$ is proper, $G \leq \operatorname{Aut}_{\mathsf{Sch}/k}(X)$ a finite subgroup of automorphisms of X defining a free, affine-admissible action. Then:*

1. *$\mathscr{L}_a$ is an invertible sheaf on $X//G$,*
2. *multiplication for the ring structure on $\pi_*(\mathscr{O}_X)$ provides an isomorphism*

$$\mathscr{L}_\alpha \otimes \mathscr{L}_\beta \to \mathscr{L}_{\alpha \cdot \beta}$$

for every pair of characters $\alpha, \beta \in \widehat{G}$,

3. *we have an isomorphism of groups*

$$\begin{aligned} &\widehat{G} \xrightarrow{\cong} \ker(\pi^* : \operatorname{Pic}(X//G) \to \operatorname{Pic}(X)) \\ &\alpha \mapsto \mathscr{L}_\alpha. \end{aligned}$$

Proof. We start by discussing the last isomorphism in the last part. By the equivalence discussed in Proposition 2, the group $\ker(\operatorname{Pic} Y \to \operatorname{Pic} X)$ identifies with the category liftings $\phi : \mu^*\mathscr{O}_X \to p^*\mathscr{O}_X$ of X's G-action on the trivial bundle $\mathscr{O}_X$. For any such action, the isomorphism $g : \mathscr{O}_X \to \mathscr{O}_X$ of $\mathscr{O}_X$-modules must of course be given by multiplication by some global section of $\mathscr{O}_X$, and since X is proper over k this must be a non-zero scalar $(\alpha(g))^{-1}$. The assignment $g \mapsto \alpha(g) \in k^\times$ is a group homomorphism by the associativity condition ϕ verifies (cfr. Definition 3). This defines a bijection between $\widehat{G}$ and $\ker(\operatorname{Pic}(X//G) \to \operatorname{Pic}(X))$ which is evidently also a group-homomorphism.
Every line bundle in $\ker(\operatorname{Pic}(X//G) \to \operatorname{Pic}(X))$ is of the form $\mathscr{L}_\alpha$, where each pair of these are isomorphic only if $\alpha = \beta$, since $\mathscr{L}_\alpha$ is quite explicitly the ring of invariants of the G-action on $\mathscr{O}_X$ defined by the character $\alpha \in \widehat{G}$.

As for the second point, we evidently have that $\mathscr{L}_\alpha \cdot \mathscr{L}_\beta$ is a subsheaf of $\mathscr{L}_{\alpha \cdot \beta}$ (when both are thought of a subsheaves of $\pi_*(\mathscr{O}_X)$), yielding a morphism of sheaves on $X//G$

$$\mathscr{L}_\alpha \otimes \mathscr{L}_\beta \to \mathscr{L}_{\alpha \cdot \beta}.$$

Checking at stalks shows that it's surjective, therefore the above morphism is an isomorphism - any surjection of line bundles is an isomorphism. ■

We're now in a position where we can finish off easily with the crux of today's talk:

Theorem 3. *Suppose X is an abelian variety. There's a bijection between*

$$\{\textit{finite subgroups } K \leq X\} \xrightarrow{1:1} \{\textit{separable isogenies } f : X \to Y\}/_{\cong}$$

Proof. If $K \leq X$ is a finite subgroup, then K acts freely on X by translations and defines an affine-admissible action as X is projective (by results discussed by Luozi's talk [7]). This implies that the quotient $X//K$ as constructed heretofore is well-defined and the corresponding quotient map $X \to X//K$ is étale. Furthermore, since on underlying topological spaces $|X//K| \cong |X|/|K|$ it follows that $X//K$ has a natural group structure, and the multiplication and inverse maps must be morphisms of varieties - indeed, if one analyses the commutative diagram

$$\begin{tikzcd} X \times_k X \arrow[r, "m"] \arrow[d, "(f\ f)"'] & X \arrow[d, "f"] \\ X/K \times_k X/K \arrow[r, "\overline{m}"] & X/K \end{tikzcd}$$

where $\overline{m}$ is the multiplication map on the quotient X/K and the only arrow above which doesn't represent a morphism of varieties a priori, we see that

$$X \times_k X \xrightarrow{f \circ m} X/K$$

is $K \times_k K$-equivariant and thus factors through (f, f) yielding a morphism of varieties, which must of course coincide with $\overline{m} \implies X//K$ is a proper smooth algebraic group which is a variety, whence an abelian variety. Conversely if $f : X \to Y$ is an isogeny whose kernel is K, then as before K acts freely and affine-admissibly on X and the map $f : X \to Y$ is of course K-invariant by construction. We thus have a morphism of varieties $h : X//K \to Y$ and a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & Y \\ X//K \arrow[ur, "h"] & \end{tikzcd}$$

h must be bijective (as this can be seen on underlying topological spaces), and separable - by the "cancellation property" of separable field extensions. Because of h's separability, the stalk of the Kahler differentials $\Omega^1_{Y/(X//K)}$ at the generic point is zero, and thus "spreads out" to the zero module on some open U of the generic point - i.e. $h : X//K \to Y$ is *generically unramified.* Since h is also 'generically flat' - cfr. section 29.27, [9] - we may assume h is also flat on U. Because k is algebraically closed and each of h's fibres are closed sets constituted by one point, it follows that h is smooth (by characterisation of smoothness in terms of fibres, as discsused in the lectures). All in all, $h\mid_U$ must be étale its bijectivity implies it must be universally injective (because k is algebraically closed); hence $h\mid_U$ is an open immersion $\implies h$ is birational. By Zariski's main theorem, h is an isomorphism and we win. ∎

Corollary 1. *A separable isogeny $f : X \to Y$ of abelian varieties X and Y is étale. Furthermore, the order of $K = \ker(f)$ equals the degree of the function field extension $k(X)/k(Y)$.*

The next corollary sets us up for the next talk, which will introduce the notion of duality in the category of abelian varities over an algebraically closed field of characteristic zero - in this setting, all isogenies are separable of course.

Corollary 2. *Suppose X and Y are abelian varieties over k and let p be k's characteristic. If $f : X \to Y$ is an isogeny whose kernel has order coprime to p, then this kernel and the kernel of the group homomorphism*

$$f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$$

are dual abelian groups.

Proof. Simply note that a field extension whose degree is coprime to their characteristic must be separable. Then apply Proposition 3. ∎

References

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